

Geodesic-Controlled Developable Surfaces for Modeling Paper Bending

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Abstract

We present a novel and effective method for modeling a developable surface to simulate paper bending in interactive and animation applications. The method exploits the representation of a developable surface as the envelope of rectifying planes of a curve in 3D, which is therefore necessarily a geodesic on the surface. We manipulate the geodesic to provide intuitive shape control for modeling paper bending. Our method ensures a natural continuous isometric deformation from a piece of bent paper to its flat state without any stretching. Test examples show that the new scheme is fast, accurate, and easy to use, thus providing an effective approach to interactive paper bending. We also show how to handle non-convex piecewise smooth developable surfaces.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism

1. Introduction

Paper is commonly used material. Therefore its shape modeling and animation is of great interest to computer graphics. We are interested in interactive modeling of paper bending, or more specifically, continuous smooth deformation of a piece of paper without creasing, while maintaining its size, i.e., allowing no size stretching or shrinking.

Under smooth bending, paper, like metal sheets, can be assumed to be inextensible, and therefore assumes the shape of a developable surface, also called a *developable*. While developable surfaces are thoroughly studied in differential geometry and a variety of numerical representations or methods have been proposed, their shape modeling with computer is still a difficult task due to the following issues:

1. The developability condition needs to be enforced, often present as a nonlinear constraint;
2. The curve of regression, i.e., the singular curve, needs to be kept out of a relevant finite patch in application;
3. The set of parameters of a chosen representation of a developable should be minimal and preferably serve as intuitive control handles in interactive applications;
4. Modeling paper bending imposes the additional requirement that the developable patch representing the paper have an isometric correspondence to the paper in its flat

state, i.e., stretching or shrinking of the paper size is not allowed.

The last requirement means that there is an isometric mapping from a bent paper to a *pre-specified* planar region, which is the same paper in its flat state. We stress that modeling a general developable surface patch only requires that the patch can be isometrically mapped to *some* plane region which is not necessarily fixed.

We present a new approach to modeling paper bending that addresses all the four issues above in a natural manner. This method is based on the fact that a developable surface is uniquely determined by a geodesic curve on it; that is to say, if we specify a smooth curve in 3D and designate it as the geodesic curve on some developable surface, then this developable surface exists and is unique, except for the case where the curve is a straight line, which is the geodesic of all the planes containing it. In fact, this developable is also easy to compute, since it is defined by the envelope of the rectifying planes of the given curve, also called a *rectifying developable*. Although the rectifying developable is well known in differential geometry, to the best of our knowledge, it has not been used in practical shape modeling applications.

The use of a geodesic to specify a developable also provides the following advantages: (1) a 3D curve is easy to

specify and manipulate, compared with many other commonly used representations of developables; (2) As we shall see, the geodesic curve offers intuitive shape control for interactively modifying the shape of a paper; (3) the use of the geodesic curve induces naturally an isometric mapping from the bent paper to its flat state, as well as allows simple computation of this mapping. These properties make this new method an effective tool for interactive paper modeling.

2. Related works

Developable surfaces. Paper sheets or metal sheets are practically inextensible or nearly so. Therefore, they are naturally represented by developable surfaces. For this reason, much research has been conducted on the computer representation of developables. One may model developables as Bézier or B-spline ruled surfaces satisfying the developability condition via nonlinear constraints [CS02, Aum04, Aum91, Aum03, CM98]. Since a developable corresponds to a curve in dual space, duality provides another method of representing a developable as a 3D curve [PW99, PF95, MBR93]. This scheme is mathematically elegant but suffers from the lack of intuitive connection between the shape of a curve in dual space and the shape of its corresponding developable. A similar idea is to use the spherical mapping of a developable surface to control its shape [Red89].

There has recently been strong interest in using developable surface for mesh segmentation and parametrization [BVI91, SH05, YGZS05, PC04]. Unfolding a surface to a plane with as little distortion as possible is also widely investigated in texture mapping, known as the problem of texture atlas generation [JKS05]. Developable surfaces have been used for fitting 3D point data [Pet04], in garment design [DJW*06], paper crafting [MS04], and architecture design [LPW*06].

Paper modeling. There are some geometric methods for paper shape modeling, taking advantage of the properties of developable surfaces. Paper creasing is considered in [KKG94, Fre04]. Yannick et al [KKG94] represent paper shape by its boundary and a mapping between boundary points is used to define rulings. This method naturally keeps singular points out of the region of the paper, but is difficult to apply to paper of general shapes. Another drawback is that the mapping between boundary points does not relate to the shape of paper in an intuitive manner.

Recovery of smooth paper shape from images is widely studied in computer vision for OCR with curled paper documents. Nail et al [GZDD06] recovers a paper shape from a contour in an image and its corresponding boundary curve in 3D by solving a system of differential equations. The vanishing curvature property of paper-like surfaces is used to recover paper structure from multiple images in [PB06]. Pilu [Pil01] recovers the planar state of a distorted image of a book page by a relaxation strategy guided by measuring isometry.

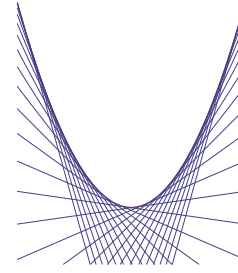


Figure 1: A 2D illustration for the concept of envelope. This example shows a parabola as the envelope of a family of straight lines in a plane.

Inextensible material simulation. Inextensible materials, like papers, metals, and clothes, can also be represented in discrete forms, e.g., triangle meshes or finite elements. Bending and stretching are simulated by following laws of elastic mechanics, using mass-spring networks. Resistance to bending is modelled by diagonal springs connected to opposite corners of adjacent mesh faces [NMK*06]. Grinspun et al [GHDS03] propose a simple technique for simulating thin-shells. Bergou et al define discrete isometric bending model which is quadratic in position, achieving efficient simulation of clothing and thin-plates [BWH*06]. There are also works using finite elements methods for un-stretchable materials simulation [Got00, NMK*06]. Zorin [Zor05] discusses the general principles of defining curvature-based energy on discrete surfaces based on geometric invariance and convergence considerations.

3. Representation of paper shape

3.1. Rectifying developable

A developable surface is defined as the envelope of a one-parameter family of planes. (An envelope of a family of manifolds is a manifold that is tangent to each member of the family at some point. Figure 1 shows the envelope of a family of lines in 2D space, which is a parabola.) The limit positions of the intersection lines of neighboring planes are on the developable surface and called *rulings*. Given a 3D curve with non-vanishing curvature, the envelope of its rectifying planes is a developable surface, called the *rectifying developable*, and the curve is a geodesic on the rectifying developable [Str61]. After unfolding the surface to a plane, the curve becomes a straight line. Figure 2 show a developable surface together with a geodesic curve on it.

In the following we give the expression of the rectifying developable surface of a given geodesic curve $p(s)$, where s is arclength. We assume that $p(s)$ has up to the third order derivative. Suppose that $p(s)$ is a geodesic on a developable surface Π . Then the principal normal direction $p''(s)$ of $p(s)$ is normal to Π at the point $p(s)$. It is known that all the points on the same ruling of a developable surface have the same

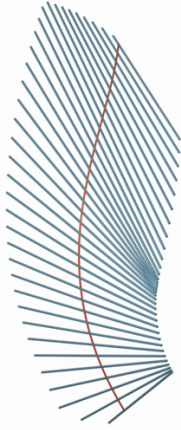


Figure 2: Rulings on a developable surface are computed from a geodesic curve on the surface, which is completely determined by any geodesic on it.

normal direction, i.e., all points on the same ruling have the same tangent plane. Therefore, $p''(s)$ is the normal to the corresponding tangent plane. Since the rulings are limit intersections of neighboring tangent planes, it can be shown that the ruling direction is $\frac{p''(s)}{|p''(s)|} \times (\frac{p''(s)}{|p''(s)|})'$ [Car76]. After simplification, we get the ruling direction $\frac{p''(s) \times p'''(s)}{|p''(s)|^2}$. The rectifying developable of $p(s)$ can therefore be expressed as

$$X(s, t) = p(s) + t(p''(s) \times p'''(s)). \quad (1)$$

The ruling direction can also be expressed in terms of the Frenet frame, as $r(s) = \tau T + \kappa B$, where τ is torsion and κ is curvature, and T and B are unit tangent vector and unit binormal vector. This vector $r(s)$ is known as the Darboux vector [Gra35], which is the instantaneous rotation axis of the Frenet frame.

3.2. Curve of regression

A general developable surface is singular along the curve of regression [PW01]. Thus it is important to know where the curve of regression exactly is and keep it out of the region of interest when a smooth surface patch is needed. Below we will derive the expression of the curve of regression in the context of a rectifying developable.

The curve of regression is the curve consisting of limit intersections of rulings. Due to the length preserving and angle preserving properties of isometry, the intersection of rulings can be computed in the plane. (Refer to Figure 3.) Let $p(s)$ be the current point and $p(s + \Delta s)$ be a neighboring point on a geodesic. Let $\bar{p}(s)$ and $\bar{p}(s + \Delta s)$ be their corresponding points on the initial flat paper in the plane. Let $\bar{q}(s)$ be the intersection point of the rulings passing through $\bar{p}(s)$ and

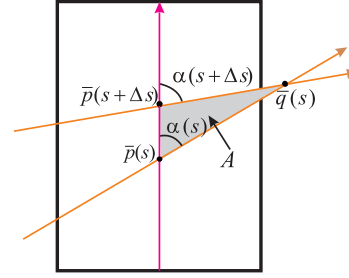


Figure 3: On a flat piece of a paper, $\bar{q}(s)$ is the intersection point of the two rulings at the points $\bar{p}(s)$ and $\bar{p}(s + \Delta s)$ on a straight line corresponding to a geodesic curve in 3D.

$\bar{p}(s + \Delta s)$. Let A be the triangle formed by $\bar{p}(s)$, $\bar{p}(s + \Delta s)$ and $\bar{q}(s)$. Let $\alpha(s)$ and $\alpha(s + \Delta s)$ be the angles between the tangent direction and ruling direction at $p(s)$ and $p(s + \Delta s)$, respectively. Since isometry is angle-preserving, these angles on the curved geodesic are equal to their corresponding angles on the initial planar paper.

By the sine law, we have

$$\frac{\sin(\alpha(s + \Delta s) - \alpha(s))}{\Delta s} = \frac{\sin(\alpha(s + \Delta s))}{l(s)}, \quad (2)$$

where $l(s)$ is a signed distance from the current point $p(s)$ on the geodesic curve to the ruling intersection point $q(s)$, or equivalently, from $\bar{p}(s)$ to $\bar{q}(s)$ (see Figure 3). By the Taylor expansion at s , $\alpha(s + \Delta s)$ can be expressed as

$$\alpha(s + \Delta s) = \alpha(s) + \alpha'(s) \cdot \Delta s + o(\Delta s). \quad (3)$$

Plugging (3) in (2) and letting Δs approach zero, we have the limit expression of $l(s)$ as

$$l(s) = \frac{\sin(\alpha(s))}{\alpha'(s)}. \quad (4)$$

The curve of regression can therefore be expressed as

$$c(s) = p(s) + l(s) \left(\frac{p''(s) \times p'''(s)}{|p''(s) \times p'''(s)|} \right) \quad (5)$$

3.3. Paper boundary and admissible shape

When a piece of paper is bent, its boundary curve also changes its shape in 3D. In this section we discuss how to compute the boundary of a piece of bent paper and use it to ensure that the bent paper takes only an admissible shape, i.e., keeping out the curve of regression. The boundary of a paper in its flat state is firstly defined and the corresponding boundary in 3D space needs to be computed. Using the isometry between the bent paper and its flat state, we will determine its boundary by explicitly computing points on it.

Recall that each ruling on the paper is associated with a point on the geodesic $p(s)$. Therefore, each ruling will intersect the boundary of the paper at two points, one on each

side of the geodesic. That is, we practically ignore all other intersections beyond these first two intersections on the two sides of the geodesic.

Let $p(s)$ be the geodesic curve on the bent paper and $r(s)$ be the ruling direction at $p(s)$. Under the isometry between the bent paper and its flat form, let $\bar{p}(s)$ be the straight line corresponding to the geodesic $p(s)$, and $\bar{r}(s)$ the corresponding ruling direction of $r(s)$ on the flat paper. Now, to compute boundary points, we start from one endpoint p_0 to sample a sequence of points along the geodesic curve $p(s)$. At each sampled position $p(s_i)$, we compute two boundary points on the ruling passing through $p(s_i)$ by the following algorithm.

1. Compute the arclength s from the start point p_0 to current point $p(s_i)$ along the geodesic. Then map $p(s_i)$ to $\bar{p}(s_i)$ on the initial plane, by measuring length s from \bar{p}_0 . Here the length preserving property of isometry is used.
2. Compute the angle between the tangent of $p(s_i)$ and the ruling direction $r(s_i)$ at $p(s_i)$. Use this angle to map the ruling direction vector $r(s_i)$ to the direction $\bar{r}(s_i)$ at $\bar{p}(s_i)$. Here the angle preserving property of isometry is used.
3. Compute intersection points \bar{c}_0 and \bar{c}_1 of $\bar{r}(s_i)$ with the given boundary of the paper in its flat state in the plane. Let l_0 and l_1 be the distances from $\bar{p}(s_i)$ to \bar{c}_0 and \bar{c}_1 . Note that l_0 and l_1 are signed distances, with $l_0 > 0$ and $l_1 < 0$, since \bar{c}_0 and \bar{c}_1 are on different sides of the geodesic.
4. Finally, the two boundary points on the bent paper are obtained by measuring distances l_0 and l_1 along the current ruling with direction $r(s_i)$ at the point $p(s_i)$.

By going through all the points on the geodesic $p(s)$, the algorithm above produces all points on the boundary of the bent region explicitly, except for the undefined regions. (See the example in Figure 4.) These uncontrolled regions are treated as planar regions and attached to the curved regions smoothly along their shared edges. This is further explained in Section 4.2.

The shape of a paper is said to be *admissible* if as a developable patch it is free of singular points except at its boundary; that is, it does not contain in its interior any points on the curve of regression. Since we have obtained the position of the curve of regression from Eqn. (4) and Eqn. (5), the condition for a piece of paper to be admissible is therefore

$$l(s) \geq l_0(s) \quad \text{or} \quad l(s) \leq l_1(s). \quad (6)$$

This is illustrated in Figure 5.

This algorithm maps every point on a paper to its corresponding point on its flat state, thus produces a development, as well as an isometric parametrization of the bent paper, which can be used for distortion-free texture mapping. See Figure 6.

3.4. Composite developable surface

So far we have shown how to represent a piece of paper using a single developable patch, i.e., with one group of rulings

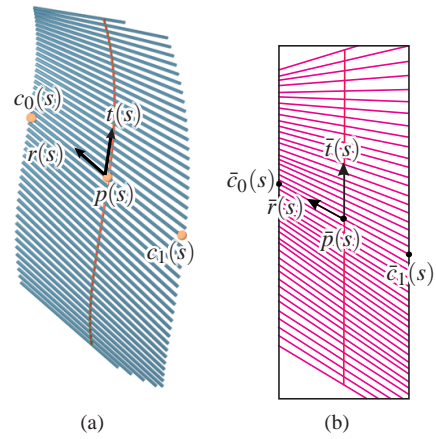


Figure 4: Boundary computation. (a) Boundary points are computed using the rulings along the geodesic curve; (b) The flat state of the paper. The two corner regions uncovered by the rulings are planar.

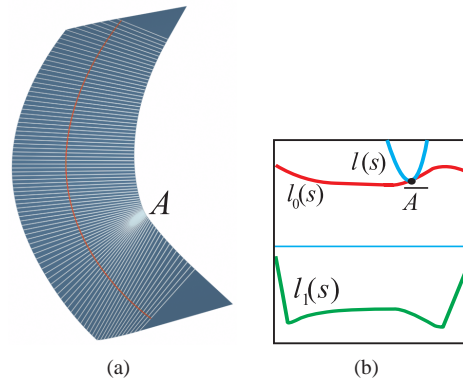


Figure 5: Condition for admissible shape. The touching point \bar{A} of $l(s)$ and $l_0(s)$ in (b) corresponds to a singular point A on the boundary of the paper in (a).

on one geodesic curve. But a piece of paper often consists of *curved developables* joined together by transition planar regions [Car76]. (See Figure 7.) Such a shape will be called a *composite developable surface*. Planar patches also appear in corner regions which are free from control by the rulings passing through a geodesic curve segment.

When a curved developable is joined with a planar piece of paper, we need to ensure that the tangent plane along the last ruling of the curved piece contains the planar piece. Note that this tangent plane is spanned by the tangent vector and ruling vector at the end point of the geodesic curve. When the paper is being bent, the shared edges between curved developables and planar regions may change. We will describe

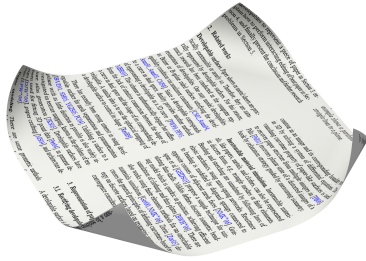


Figure 6: A textured paper as a composite developable surface. The bent paper is isometric to its flat state, which is used here as the domain for distortion-free texture mapping.

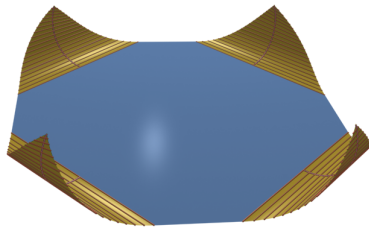


Figure 7: Here we show the curved parts and planar part of the composite paper in Figure 6.

a method for ensuring smooth connections of different regions in Section 4.2.

4. Interactive modification

In a paper editing session we first define the boundary of paper in its flat state and its structure, i.e., the arrangement of planar regions and curved regions. Each curved region is assigned a traversal straight line as the designated geodesic. Then a composite developable surface is generated in real time manner from a geodesic curve that the user specifies interactively in 3D. By controlling the specified geodesic curve, the paper shape will change accordingly. Hence, this geodesic is called the *control geodesic*. Control geodesics are represented as polynomial curves in our experiments; we use Bézier curves and B-spline curves.

4.1. Editing tools

4.1.1. Control points

When using a Bézier or B-spline curve as the control geodesic, the shape of the geodesic curve can be modified directly by moving their control points, and that causes the paper to change its shape accordingly. This provides intuitive shape control. See Figure 8.

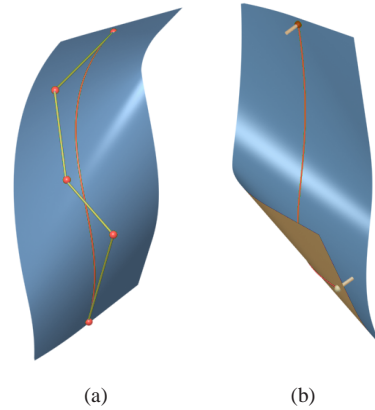


Figure 8: Shape editing. (a) The shape is modified by moving the control points of a geodesic Bézier curve; (b) The shape is modified by controlling the two handles defined at two endpoints of a geodesic curve. A handle is normal to the surface.

4.1.2. Control handles

While the control points of the Bézier geodesic curve is intuitive for editing a developable surface, it is still quite different from the human experience of paper bending. Normally, a piece of paper is bent with two hands hold at two positions of the paper. This motivates us to consider how to use two control handles to simulate this operation.

In this mode a geodesic curve still serves as the underlying control representation. But now they are computed from interpolating the positions and orientation vectors at the two ends, which are specified by the two control handles. Each control handle consists of a *position handle* and a *normal handle*; the position handle specifies the endpoint of the geodesic and the normal handle specifies a direction parallel to the principal normal vector of the geodesic at the end point, which is also normal to the developable to be generated.

When the control handles are modified, a new interpolating control geodesic is determined by computing control points of a Bézier curve or B-spline curve representing the geodesic. We use a Bézier curve of degree 4 or higher in our experiments.

When a Bézier curve is used as control geodesic, the curve is expressed as $P(t) = \sum_{i=0}^n P_i B_{i,n}(t)$, $t \in [0, 1]$, where P_i are control points and n is the degree. Control handles are defined at the end control points of the Bézier curve. Thus the position control is easily realized by the Bézier curve interpolating its end control points. The principle normal direction of $P(t)$ is $N(t) = (P'(t) \times P''(t)) \times P'(t)$. Let $N_0 = N(0)$ and $N_1 = N(1)$ denote the principle normals at the two endpoints. Then N_0 and N_1 can be expressed in terms of control

points [Far96] as follows

$$\begin{aligned} N_0 &= \lambda(P_1 - P_0) \times (P_2 - 2P_1 + P_0) \times (P_1 - P_0), \\ N_1 &= \lambda(P_n - P_{n-1}) \times (P_n - 2P_{n-1} + P_{n-2}) \times (P_n - P_{n-1}), \end{aligned}$$

where $\lambda = n^3(n-1)$.

Let \bar{N}_0 and \bar{N}_1 be the desired unit normal vectors specified interactively at the two endpoints. Then the constraints for the new control geodesic curve are

$$\begin{aligned} N_0 &= a_0 \cdot \bar{N}_0 \\ N_1 &= a_1 \cdot \bar{N}_1 \end{aligned} \quad (7)$$

for some nonzero constants a_0 and a_1 .

The above constraints (7) in general cannot determine a relatively high degree interpolating Bézier curve uniquely. So we solve for the interpolating curve by minimizing the following objective function, in order to ensure the continuity of the control geodesic curve with respect to the continuous input control:

$$f(\Delta X) = \left| \frac{N_0}{|N_0|} - \bar{N}_0 \right|^2 + \left| \frac{N_1}{|N_1|} - \bar{N}_1 \right|^2 + \lambda \cdot |\Delta X|^2,$$

where $\Delta X = (\Delta P_1, \dots, \Delta P_{n-1})$ are the increments to the control points. The $|\Delta X|^2$ term is used to ensure a gradual change of the geodesic curve. Its coefficient λ is set to 0.1 in our experiments.

We minimize $f(\Delta X)$ using the optimization module in the IMSL library, which is a quasi-Newton method [IMS]. Although we do not always obtain exact interpolation to the desired normals \bar{N}_0 and \bar{N}_1 , the performance is satisfactory as a real time interactive shape editing tool. (See Figure 8(b) for an example.) This algorithm is incremental and the result of each update depends on the previous state.

4.1.3. Curve length preserving

Since our method controls the shape of paper by modifying a curve on it, we need to preserve the length of the curve during interactive manipulation. Modifying a curve with its length preserved is a non-trivial task. We use the a simple strategy with Bézier curves to control geodesic curves to approximately achieve length preservation.

Suppose that a geodesic curve is modified by the user to give the current geodesic $P(t)$, $t \in [0, 1]$. We start from the endpoint $P(0)$ of the geodesic to measure a fixed length using numerical integration. Suppose that we stop at point $p(t_1)$. Then we find the Bézier control points of the curve segment $P(t)$, $t \in [0, t_1]$, and reparameterize it to be the new geodesic control curve, denoted by $\hat{P}(\hat{t})$, $\hat{t} \in [0, 1]$. Because the user normally makes continual and incremental change to the end positions and normals, there is only a slight difference between the exact interpolating curve $P(t)$, $t \in [0, 1]$, and the modified curve $\hat{P}(\hat{t})$, $\hat{t} \in [0, 1]$. This change is usually small and within acceptable level in our experiments.

After the above reparameterization, the control points of

the curve will be shifted in general. Let P_i be the control point being edited or dragged. Suppose that P_i is shifted to P'_i due to the reparameterization. Then we translate the entire curve $\hat{P}(\hat{t})$ by $P_i - P'_i$ so that the new position of P_i is always preserved, giving the natural impression that the curve is being dragged by the change of P_i , while preserving its length.

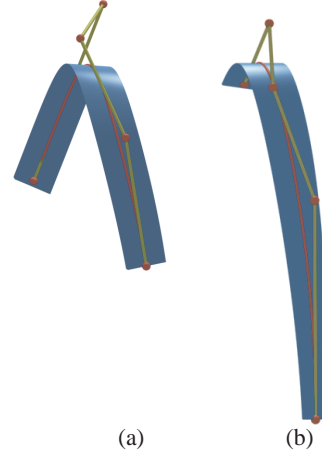


Figure 9: Curve length preserving. The strip in (b) is obtained by dragging the lowest control point of the strip in (a). The length of the geodesic curve is preserved.

4.2. Modifying composite shape

A composite developable shape contains both curved and planar regions joined together smoothly. A piece in a composite developable surface does not change its type (i.e., planar or curved) during editing process. As shown in Figure 10, when modifying a geodesic curve on the curved piece A, the shape of its neighboring planar piece B changes accordingly; but B is always on the tangent plane along A's end ruling which abuts B. The updated shape of B is determined from the structure of paper's flat state and the changed rulings of A. So we just need to wield the polygon B to its shared edge with A using a rigid transformation. Other regions connected to B are then transformed by the same transformation. Figure 11 shows an example of a composite developable with two curved pieces controlled by moving Bézier control points.

4.3. Admissible shape during editing

A condition for a single developable piece to be admissible is presented in Section 3.3. For a composite piece, in addition to constraining each piece to be admissible, the rulings of different pieces are not allowed to intersect each other inside the paper region. During shape editing, we check the condition (6) after modification for each developable piece and prevent rulings of different pieces from intersecting each

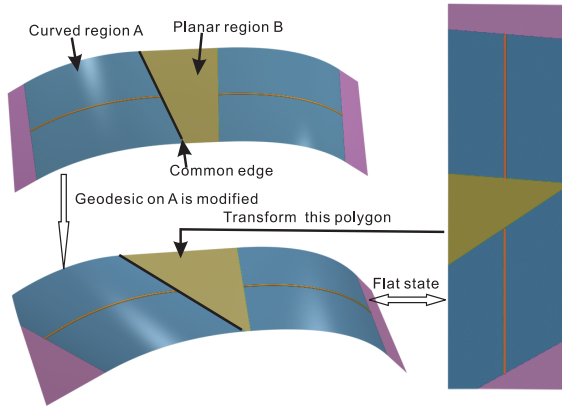


Figure 10: Editing a composite developable shape. After geodesic on curved region A is edited, the boundary of its neighboring planar region B is obtained from paper's flat state and is smoothly attached to A at their shared edge.

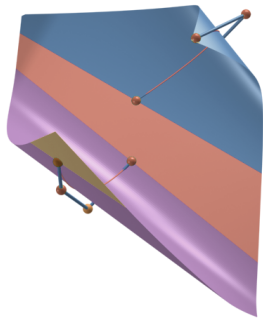


Figure 11: Editing a composite developable shape. There are two curved pieces, each being determined by a Bézier geodesic curve generated by control point manipulation.

other inside the paper region. Note that this latter intersection test only needs to be performed for the end rulings of different developable pieces.

5. Examples and Extension

Using the proposed representation of geodesic-controlled developables, we can model papers of different shapes, concave or even with holes, as well as closed paper strips. See the examples in Figure 12. These examples are designed by first defining their planar structure and geodesic curves on their planar counterparts and then manipulating the control geodesics. Figure 15 and Figure 16 show a desk, a chair and an architectural model designed with our method.

Given an initial developable surface, it is possible to specify a geodesic curve on it and animate the surface by directly

editing this curve, i.e., without explicitly referring to the paper's flat state. Such a test example is shown in Figure 14. Here, we select a circle (i.e., a geodesic) on a circular cylinder and deform the cylinder by modifying the circle into different shapes. The length of the control curve is preserved by scaling.

The idea of control geodesics can be extended to non-geodesic curves. With an arbitrary curve $p(s)$ on a developable surface which is not a geodesic curve, its corresponding curve $\bar{p}(s)$ under isometry in the plane is no longer a straight line. Now we are going to find a unique developable containing $p(s)$ such that the geodesic curvature of $p(s)$ on Π is the same as the curvature of the planar curve $\bar{p}(s)$ (which is also the geodesic curvature of $\bar{p}(s)$), i.e., meeting the requirement that the geodesic curvature be preserved during paper bending.

It is well known that the principal normal $p''(s)$ can be decomposed into the direct sum of the geodesic curvature vector $g(s)$ and the normal curvature vector $n(s)$; and these three vectors are in the plane normal to the tangent vector $p'(s)$. Recall that the geodesic curvature is preserved by isometry. Suppose that we know the curvature $c(\bar{p}(s))$ of $\bar{p}(s)$, which can easily be computed. Then we have $|g(s)| = c(\bar{p}(s))$. Now, knowing the value $|g(s)|$ and the curvature vector $p''(s)$, we can easily find the normal curvature vector $n(s)$. Then the developable surface Π to be constructed is the envelope of the planes passing through $p(s)$ and having $n(s)$ as their normal vectors. An example generated using this technique is shown in Figure 13, where the selected control curve is a circular arc adapting to the shape of the paper strip.

Using an arbitrary curve on a developable as a control curve has certain advantages when it is difficult to find a geodesic curve on the developable. Clearly, the use of a geodesic curve is a special case where the geodesic curvature is zero. However, we note that when using an arbitrary planar control curve $\bar{p}(s)$, the curvature $|p''(s)|$ of the modified curve $p(s)$ is required to be larger than the geodesic curvature of $\bar{p}(s)$ for every s . This leads to the interesting problem to constraining the control curve $p(s)$ to have bounded curvature during shape editing. We leave this as a problem for future research.

6. Discussions and Future works

Using a geodesic curve to control the shape of a developable surface has many advantages, including intuitive control and ease of preventing singular points in the region of interest. We have shown that this approach is suitable for bending paper or other non-stretchable materials in interactive applications. In this context, we will further investigate the representation of the control geodesic curve with length preserving property. In our current implementation, numerical integration is used to modify the control geodesic curve to make its length constant. We believe the 3D PH curve holds

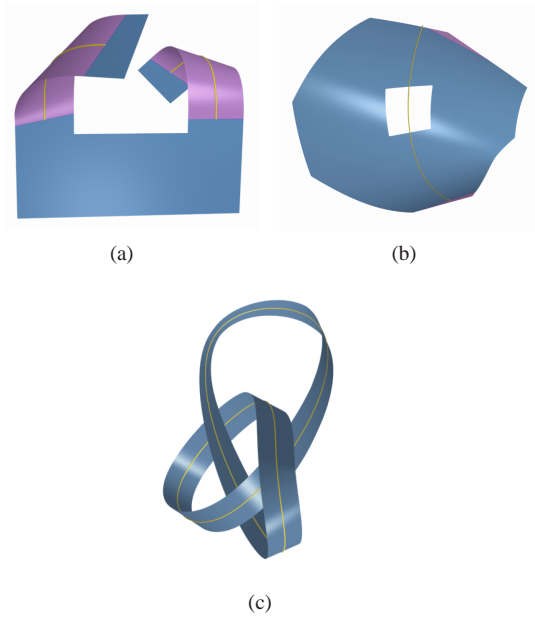


Figure 12: Developable surfaces generated from geodesic curves. Different colors are used to illustrate different pieces, i.e. curved developables and planar pieces. (a) A local control example; (b) a paper with hole on it; (c) a closed paper strip.

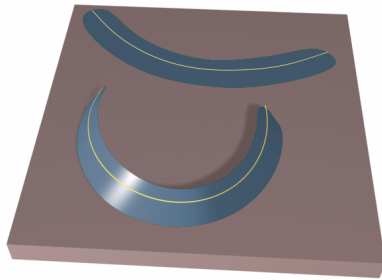


Figure 13: Editing a developable surface using an arbitrary curve on it. In this example, the planar curve is a circular arc.

the promise of providing an elegant solution to this problem [FH03].

Our technique is based on the well known concept of rectifying developables and it is the first time this representation of developables is applied to shape modeling and design. We hope this representation can find more applications in shape modeling. Some promising applications include surface unfolding [SE06] and using developable surface to fit some nearly developable target shapes [Pet04].

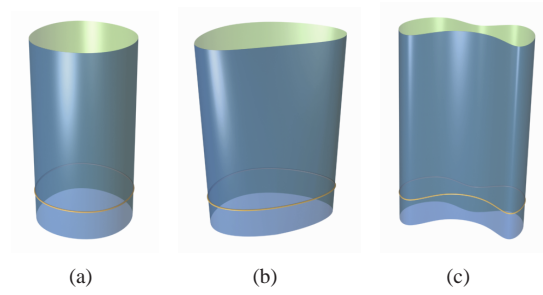


Figure 14: Deformation of a tube. We deform the shape of a cylinder by first specifying a circle on it and modifying the circle.

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Figure 15: A desk and a chair designed with our methods. The desk model is composed of four curved developables as legs and a transition planar region as the desktop. The chair model is made of developable surfaces with concave boundary and holes.

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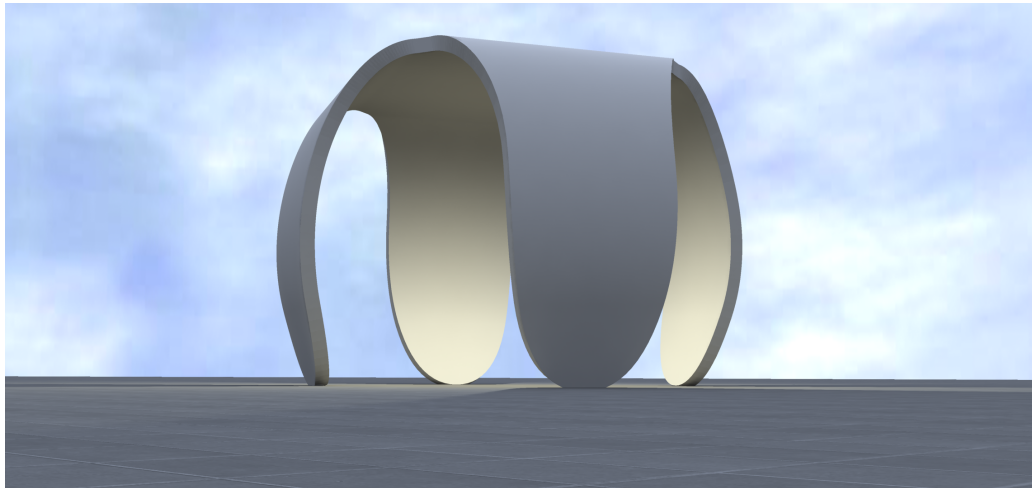


Figure 16: An architectural model.

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